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Delay in a soliton transmission across an interface between two Toda lattices

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Abstract

The transmission of a single soliton is investigated numerically across an interface between two Toda lattices, which are connected by a harmonic lattice. The soliton transmission coefficient is used as a measure of transmission. When the spring constant (κ) of the harmonic spring is small and the number of harmonic springs is greater than or equal to 2, a delay in the transmission of the soliton is found for proper κ. It is shown that the delay in the soliton transmission is due to the existence of the quasi-localization of the wave in the harmonic lattice and the agreement of the time scale of the motion between the two springs.

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1. Introduction

In the past few decades, various aspects of the soliton have been investigated and the application of the soliton has been developed in many fields [1–7]. It is an important problem in the study of the basic properties of soliton propagation and in practical applications that a soliton is scattered by an interface between nonlinear media, which support the soliton propagation. For example, in the realistic optical soliton communication system, solitons are propagated through several fibre segments joined by a fibre splice [8]. When a soliton passes an interface between two segments, the soliton is scattered, causing its transmittance to shrink.

Anderson *et al* [9] studied soliton tunnelling to improve an optical soliton compression using connected fibres. When a soliton passes an interface between two fibres, the transmitted pulse contains nonsoliton waves, leading to degradation of the soliton [10, 11]. It was shown that the nonsoliton waves can be decreased by using a proper junction of a finite-length fibre. Kubota and Odagaki [12] studied the soliton transmission across an interface between two

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identical Toda lattices connected by a harmonic lattice. The resonant transmission of the soliton was found by controlling the spring constant of the harmonic lattice.

We treat the Toda lattice as a representative nonlinear medium which supports the propagation of soliton. The reasons that we treat the Toda lattice are as follows: (1) since the Toda lattice is a discrete system, the propagation of solitons can be studied numerically without discretization of the space coordinate. (2) Since the exact solution exists for the Toda lattice without recourse to perturbation expansions or continuum approximations, we can change the wave number of the soliton as large as we want. (3) Since the Toda lattice can be reduced to the KdV equation by perturbation expansions or continuum approximations [13], the results of the Toda lattice can be compared with those of the KdV equation. It should be noted that the Toda lattice can be realized by a nonlinear LC ladder-type circuit [14–17] and our numerical results can be tested experimentally [12].

In the present paper, we numerically investigate the soliton scattering by the interface, using the same model in our recent work [12]: two identical Toda lattices are connected by a harmonic lattice. In [12], the number of springs (denoted by N) in the harmonic lattice is fixed at $N = 1$. We investigate the transmission of the soliton for $N \geqslant 2$ and have found an anomalous transmission of the soliton.

The harmonic lattice does not support the propagation of soliton and thus can be interpreted as a coupling impurity. Most of the studies on the effects of impurities on the soliton propagation are based on continuous media [18–21]. There are few studies for the case of a coupling impurity on discrete systems [22]. For a weak coupling impurity, the transmittance of the soliton has a minimum as a function of the wave number of the incident soliton. The soliton was not much affected by a strong coupling impurity [22]. It is well known that a harmonic lattice with a light mass impurity or a strong coupling impurity has a localized mode [23]. Many numerical [22, 24–26] and analytical studies [25, 26] have shown that localized waves can exist at a light mass impurity or a strong coupling impurity in a nonlinear lattice. The frequency of the localized wave in the Toda lattice decreases with the increase in the amplitude of the localized wave [24, 26].

The present paper is organized as follows. Our model is introduced in section 2, where we define basic quantities which are used in the following discussion. In section 3, we numerically investigate the transmission profile of the soliton. When $N \geqslant 2$ and the spring constants of the harmonic springs are chosen properly, we find that the majority of the incident soliton stays in the harmonic lattice for a while and then a large soliton leaves the harmonic lattice (delay in the soliton transmission). In section 4, we discuss the origin of the delay in the soliton transmission by a comparison of the time scale of the motion between two springs. We summarize the results of the present study in section 5.

2. Model

We consider two identical Toda lattices [27] connected by a harmonic lattice. We use dimensionless variables defined in [12]. The Hamiltonian is given by

$$H = \sum_n \left[\frac{P_n^2}{2} + \Phi_n(U_n) \right], \tag{1}$$

$$U_n = Q_{n+1} - Q_n, \tag{2}$$

where P_n and Q_n are the dimensionless momentum and displacement of a particle on site n, respectively. Φ_n is the interaction potential energy between the particles on sites n and $n + 1$

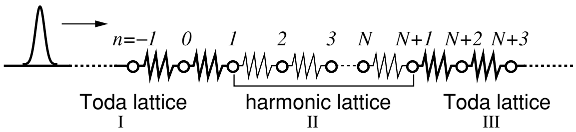

Figure 1. The perspective of an incident soliton on two Toda lattices connected by a harmonic lattice (named region II). Regions I and III are identical Toda lattices.

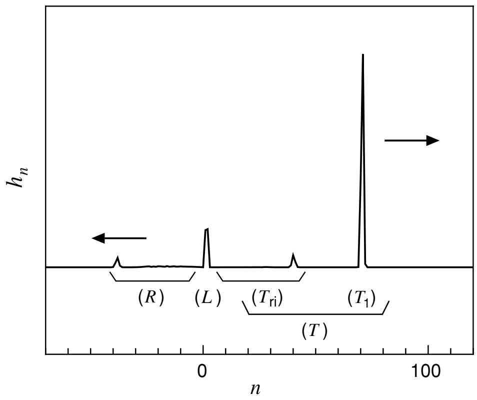

Figure 2. The profile of waves, after some time the soliton passes the harmonic lattice, which consists of transmitted (T), reflected (R) and localized (L) waves. The transmitted waves consist of a large soliton (T_1) and small waves (T_{ri}). The vertical axis is energy density h_n normalized by the energy of the incident soliton.

given by

$$\Phi_n(U_n) = \begin{cases} [\exp(-U_n) + U_n - 1] & \text{for} \quad n \leqslant 0 \text{ and } n \geqslant N+1, \\ \frac{1}{2}\kappa U_n^2 & \text{for} \quad 1 \leqslant n \leqslant N, \end{cases} \tag{3}$$

where N denotes the number of harmonic springs and κ is a spring constant of the harmonic springs. The equation of motion is given by

$$\frac{\mathrm{d}^2 Q_n}{\mathrm{d}\tau^2} = -\frac{\mathrm{d}}{\mathrm{d}Q_n}[\Phi_n(U_n) + \Phi_{n-1}(U_{n-1})], \tag{4}$$

where τ is the dimensionless time.

At time $\tau = 0$, we prepare a single soliton

$$U_n = -\ln\{1 + \omega_0^2 \operatorname{sech}^2[k_0(n - n_0) - \omega_0\tau]\}, \tag{5}$$

$$\omega_0 = \sinh k_0, \tag{6}$$

as an incident wave (see figure 1), where k_0 is regarded as a wave number of the soliton and n_0 denotes the location of the soliton at $\tau = 0$. We set $n_0 \ll 0$ so that the incident soliton is far left from the harmonic lattice.

When the incident soliton passes the harmonic lattice, the incident wave is divided into three waves: transmitted, reflected and localized waves (see figure 2). The vertical axis in figure 2 is the normalized energy density given by

$$h_n(\tau) = \frac{1}{2E_0}\{P_n^2(\tau) + \Phi_n[U_n(\tau)] + \Phi_{n-1}[U_{n-1}(\tau)]\}, \tag{7}$$

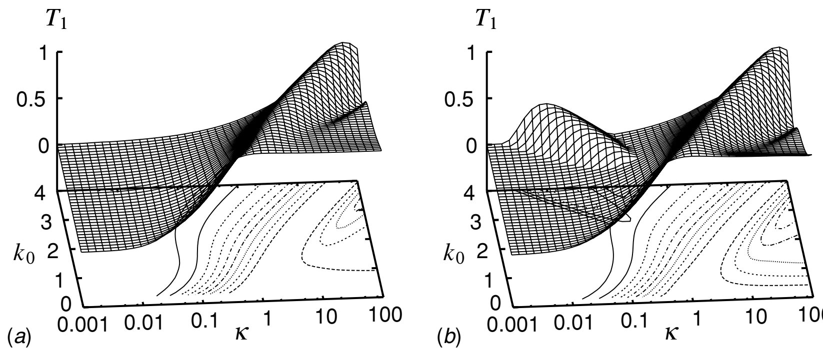

Figure 3. The soliton transmission coefficient T_1 on the (k_0, κ) domain for the number of harmonic springs (*a*) $N = 1$ and (*b*) $N = 2$. The contour spacing of the contour lines is 0.1.

where E_0 is the energy of the incident soliton, given by $E_0 = 2(\sinh k_0 \cosh k_0 - k_0)$. The transmitted wave consists of a 'large soliton' and many small waves. Most of the energy of the transmitted wave is carried by the large soliton. We can use the *soliton transmission coefficient* [28, 29] given by

$$T_1 = \frac{E_1}{E_0}, \tag{8}$$

as a measure of transmission, where E_1 is the energy of the 'large soliton'. We call the soliton at the front of the transmitted wave a 'frontier soliton'. It should be noted that in most cases, the 'large soliton' is the 'frontier soliton'.

3. Numerical results

We integrate equation (4) numerically using a third-order bilateral symplectic algorithm [30]. The setting of the numerical integration is the same as used as in [12].

3.1. κ dependence of transmission coefficient T_1

Transmission coefficient T_1 was obtained for various spring constants κ of the harmonic spring and the wave number k_0 of the incident soliton. In figure 3, we show T_1 on the (k_0, κ) domain for (a) $N = 1$ and (b) $N = 2$. As shown in [12], figure 3(*a*) shows that T_1 has a maximum as a function of κ for each k_0 and the maximal value is $T_1 \simeq 1$ (resonant transmission of the soliton). Figure 3(*b*) shows that for large k_0 ($\gtrsim$2.5), T_1 has two local maxima as a function of κ: for large κ ($\gtrsim$1) and for small κ ($\lesssim$0.2). For convenience, we call the second local maximum for small κ a 'small hump'. As we show below, the small hump is due to an anomalous transmission of the soliton, i.e., the large soliton trapped in the harmonic lattice for a while.

It should be emphasized that the small hump also appears for $N \geqslant 3$ and does **not** appear for $N = 1$. We obtained T_1 for $N = 3, 4, 5, 10$ and 50, and the small hump appears for $\kappa < 1$ and $k_0 \gtrsim 2.5$.

We summarize the profile of the small hump as follows: the small hump appears for $N \geqslant 2$, $\kappa < 1$ and $k_0 \gtrsim 2.5$. Contrary to the resonant transmission of the soliton, the top of the small hump shifts to smaller κ with the increase in k_0. We note that the small hump is not

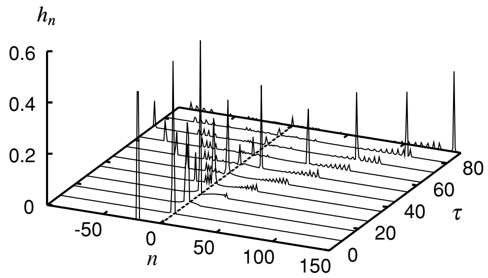

Figure 4. The spacetime evolution of the energy density $h_n(\tau)$ normalized by total energy for $N = 2$, $k_0 = 3.5$ and $\kappa = 0.03$. τ is the dimensionless time. Note that a large soliton passes small waves after staying for a while in the harmonic lattice.

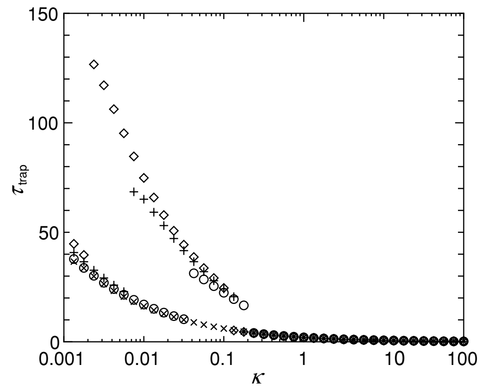

Figure 5. The trapping time τ_{trap} of the large soliton in the harmonic lattice as a function of κ for $k_0 = 2.5$ ($\times$), $k_0 = 3.0$ ($\circ$), $k_0 = 3.5$ (+) and $k_0 = 4.0$ ($\Diamond$). The number of harmonic springs is $N = 2$.

due to the second resonance of the resonant transmission, and the second resonance exists for $N \geqslant 1$ and large κ ($\gtrsim$100).

3.2. Temporal evolution of wave

In figure 4, we show the normalized energy density $h_n(\tau)$ as a function of n and τ for $N = 2$, $k_0 = 3.5$ and $\kappa = 0.03$. We note that this set of parameters corresponds to the top of the small hump. This figure shows that at $\tau \sim 20$, small solitons leave the harmonic lattice and travel in region III. The majority of the incident wave remains in the harmonic lattice for a while. At $\tau \sim 50$, the majority of the remaining wave leaves the harmonic lattice and travels in region III as a large soliton. The large soliton overtakes the small solitons that leave the harmonic lattice earlier.

This phenomenon is a delay in the transmission of the soliton. In figure 5, we show the trapping time τ_{trap} of the large soliton in the harmonic lattice as a function of κ for $k_0 = 2.5$, 3.0, 3.5 and 4.0. This figure shows that for $k_0 \geqslant 3.0$, τ_{trap} jumps at κ, which corresponds to the edges of the small hump of T_1.

In figure 6, we obtain the temporal evolution of the relative displacement U_n for $n = 2$ (the harmonic spring) and $n = 3$ (the Toda spring) for the same parameters as in figure 4. It

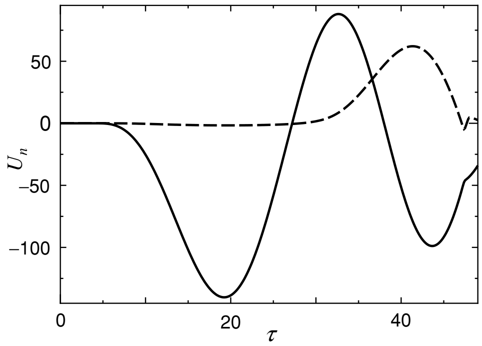

Figure 6. Temporal evolution of the relative displacement U_n for $n = 2$ (——, the harmonic spring) and for $n = 3$ (– –, the Toda spring) for $N = 2$, $k_0 = 3.5$ and $\kappa = 0.03$.

should be emphasized that the evolution of U_3 is mainly determined by U_2 since U_4 is initially at rest. At $\tau \simeq 19$, the Toda spring (U_3) is slightly compressed and, as a result, small waves appear in region III. After that, the Toda spring follows the motion of the harmonic spring with some delay. The Toda spring stretches appreciably and then compresses quickly. At $\tau = 47.5$, U_3 is less than zero and, as a result, the large soliton appears in region III. At this time, U_2 almost returns to 0 and the energy in the harmonic lattice is almost transported to region III.

Figure 6 also shows that the temporal evolution of U_2 can be regarded as a sine function between $\tau \simeq 15$ and 47.5. We set the phase of the sine function to be 0 at $\tau \simeq 27$ (i.e., $U_2 = 0$ and $\mathrm{d}U_2/\mathrm{d}\tau > 0$). When the large soliton leaves the harmonic lattice, the phase of U_2 is between $3\pi/2$ and 2π. This phase of U_2 strongly relates to the emergence of the large soliton, i.e., the phase of U_2 satisfies the condition, so that U_2 almost returns to 0 and U_3 compresses quickly.

4. Analysis and discussion

In section 3.2, we observed the temporal evolution of the wave and obtained several conditions for the delay in the soliton transmission. In this section, we estimate the parameter region for the delay in the soliton transmission from those conditions.

4.1. Quasi-localization

As described in section 3.2, for the delay in the soliton transmission, the majority of the incident soliton stays in the harmonic lattice for a while, i.e., localization occurs temporarily. For that, the system supports the existence of a temporary localization in the harmonic lattice. The energy of the reflected waves and of the frontier soliton should be sufficiently small so that the majority of the incident wave remains in the harmonic lattice.

4.1.1. Existence of quasi-localization. For $N \geqslant 2$, as can be seen from figure 6, the harmonic spring U_2 oscillates considerably, i.e., the temporary localization can exist in the harmonic lattice for $\kappa \ll 1$. As shown in [12], for $N = 1$, no localized waves exist in the harmonic lattice for $\kappa < 1$. As a result, the delay in the soliton transmission does not exist when $N = 1$.

These facts can be explained by considering localized modes in a harmonic lattice with a sequence of weak coupling impurities. When the number of weak coupling impurities is

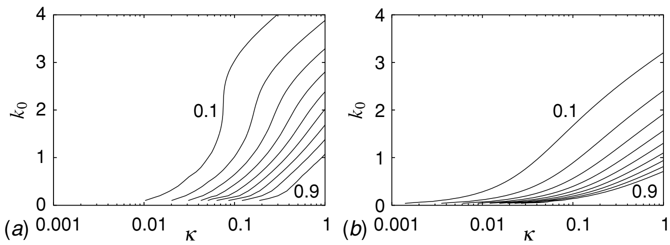

Figure 7. (*a*) The contour lines for the energy of the frontier soliton normalized by the incident one for $N = 2$ on the (k_0, κ) domain. (*b*) The contour lines of the approximants determined by equations (9) and (10) on the (k_0, κ) domain. The contour spacing is 0.1.

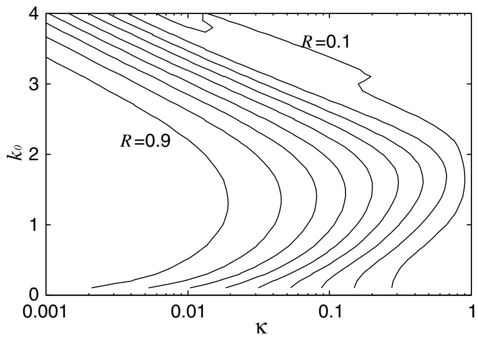

Figure 8. The contour lines of the reflected energy R for $N = \infty$ on the (k_0, κ) domain. The contour spacing is 0.1.

$N' = 1$, there are no localized modes. On the other hand, for $N' \geqslant 2$, there are quasi-localized modes, where particles in the impure region oscillate almost independently of the motion of particles outside the impure region.

4.1.2. Energy of frontier soliton. We obtained the energy of the frontier soliton (denoted by T_f) normalized by the energy of the incident soliton for $N = 2$ and plot it in figure 7(*a*). This figure shows that for small k_0 ($\lesssim 1$) and large κ ($\gtrsim 0.1$), T_f is large. T_f decreases with the decrease in κ.

4.1.3. Reflected energy. In order to show the energy of the reflected waves scattered by the entrance of the harmonic lattice, we obtained the reflected energy (denoted by R) for $N = \infty$ normalized by the energy of the incident soliton and plot it in figure 8. This figure shows that R is small for large κ ($\gtrsim 0.1$) except for intermediate k_0 ($1 \lesssim k_0 \lesssim 2$). But, for large κ and small k_0, the energy of the frontier soliton is *not* small, as described in section 4.1.2. As a result, a delay in the soliton transmission can exist for large κ and large k_0. Figure 8 also shows that for fixed k_0, R increases with the decrease in κ and, as a result, the delay in the soliton transmission shrinks for rather small κ.

4.2. Two waves

As described in section 3.2, for the delay in the soliton transmission, the small frontier soliton leaves the harmonic lattice at first and then the large soliton leaves. Here, we consider the origin of the smallness of the frontier soliton and the condition for the emergence of the large soliton.

4.2.1. Frontier soliton. In this section, we estimate the energy of the frontier soliton by considering the motion of the particles around the boundary between regions II and III, so as to understand the transmission of the wave for small κ.

For $\kappa \ll 1$, the time scale of the motion of the harmonic spring is much longer than that of the compressed Toda spring. The particles at $n \geqslant N+1$ are assumed to move in a quasi-stationary manner. We have

$$-\kappa U_N - \exp(-U_{N+1}) + 1 = 0. \tag{9}$$

When the incident soliton comes near $n = N$, the particle at $n = N$ moves to the right and compresses the harmonic spring. When the particle at $n = N$ stops, we estimate the relative displacement U_{N+1} (the Toda spring) as follows. For simplicity, we consider the energy only in $n = [N, N+2]$ and approximate its energy by half of the incident energy $E_0/2$. Since the particles at $n = N+1$ and $N+2$ move quasi-stationary, the kinetic energy of these particles can be ignored. Therefore, we have

$$\tfrac{1}{2}\kappa U_N^2 + \exp(-U_{N+1}) + U_{N+1} - 1 = \sinh k_0 \cosh k_0 - k_0. \tag{10}$$

From equations (9) and (10), we obtain U_{N+1} numerically. We approximate the amplitude of the frontier soliton by U_{N+1} and finally obtain the wave number of the frontier soliton for various k_0 and κ. We show the energy T_f of the frontier soliton normalized by E_0 in figure 7(*b*).

This figure shows that T_f decreases with the decrease in κ and the numerical results (figure 7(*a*)) are qualitatively described. For small κ, the dependence of T_f on k_0 and κ can be qualitatively explained by taking into account the quasi-stationary motion of the Toda spring.

4.2.2. Large second soliton. As described in section 3.2, when the large soliton leaves the harmonic lattice, the phase of U_n at the end of the harmonic lattice ($n = N$) is between $3\pi/2$ and 2π regarding the temporal evolution of U_N as a sine function.

In order to understand the delay in the soliton transmission, we compare the periods of two oscillators: the oscillator consists of a particle connected to a wall by the harmonic or Toda spring, and the mass of the particle is set at 1. The half-period of the harmonic oscillator is given by

$$\tau_{\text{harm}}(\kappa) = \frac{1}{2}\frac{2\pi}{\sqrt{\kappa}}. \tag{11}$$

In the case of the Toda spring, we divide the period of oscillation at the time when the relative displacement is $U = 0$ into two parts: a 'stretching half-period' and a 'compressing half-period'. Here we consider the 'stretching half-period'. The stretching half-period τ_{stre} depends on the energy E_{os} of the oscillation. For large E_{os}, the stretching half-period τ_{stre} can be approximated by[2]

$$\tau_{\text{stre}} = \sqrt{8E_{\text{os}}}. \tag{12}$$

After the frontier soliton left the harmonic lattice, the energy in the harmonic lattice is less than E_0, i.e., $E_{\text{os}} = \alpha E_0$, where α is a real number in [0, 1]. We approximate $\alpha = 0.2$ determined from the numerical results on the energy density.

[2] The equation of motion for the oscillator of the Toda spring is given by $\ddot{Q} = \exp(-Q) - 1$. For $Q \gg 1$, the equation is well approximated by $\ddot{Q} = -1$. We integrate this equation with an initial condition: $Q = 0$ and $\dot{Q} = v_0$. We obtain $Q(\tau) = -\tau^2/2 + v_0\tau$ and $\tau_{\text{stre}} = 2v_0 = \sqrt{8E}$.

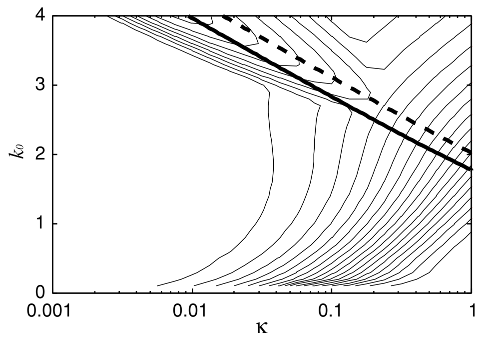

Figure 9. The thick curves denote the parameters (k_0, κ) given by equation (14) with $\theta_{\text{harm}} = 3\pi/2$ (——) and 2π (– –). Here, θ_{harm} denotes the phase of the harmonic spring at $n = N$ when the large soliton leaves the harmonic lattice. The thin solid curves denote the contour lines of the soliton transmission coefficient T_1, obtained in section 3.1.

The results in section 3.2 mean that the phase of U_N (harmonic spring) varies from 0 to θ_{harm} during τ_{stre}, i.e.,

$$\tau_{\text{stre}}(k_0) = \frac{\theta_{\text{harm}}}{\pi}\tau_{\text{harm}}(\kappa). \tag{13}$$

From equations (11), (12) and (13), we finally obtain

$$\kappa = \frac{\theta_{\text{harm}}^2}{16\alpha(\sinh k_0 \cosh k_0 - k_0)}. \tag{14}$$

The numerical results in section 3.2 show that θ_{harm} is between $3\pi/2$ and 2π at the top of the small hump of T_1. We compare equation (14) with the contour lines of T_1 in figure 9. This figure shows that the top of the small hump of T_1 is between the curves determined by equation (14) with $\theta_{\text{harm}} = 3\pi/2$ and 2π.

From equation (13), θ_{harm}/π can be regarded as the ratio between the time scales of the motion of the two springs. The ratio between the time scales of the motion of the two springs is important for the delay in the soliton transmission. As a result, the location of the small hump of T_1 shifts to smaller κ with the increase in k_0.

4.3. Summary of analysis

In section 3.2, we observed the temporal evolution of the waves for parameters corresponding to the small hump of T_1, where the delay in the soliton transmission occurs. We obtained the four conditions for the delay in the soliton transmission as follows: (1) the energy of the frontier soliton is small. (2) The quasi-localization can exist in the harmonic lattice. (3) The energy of the reflected waves is small, so that the amplitude of the quasi-localized wave is large. (4) The phases of the two springs match, so that the Toda spring at the edge of region III compresses quickly.

From these conditions, we estimated the parameter region for the delay in the soliton transmission and compare it with the contour lines of T_1 in figure 10. This figure shows that the small hump of T_1 is in the region that is approximately determined from the conditions (1)–(3). This figure also shows that the position of the small hump is well represented by equation (14) with $\theta_{\text{harm}} = 7\pi/4$, which is derived from the condition (4). Moreover, it is described in section 4.1.1 that the number of harmonic springs is $N \geqslant 2$ from the condition (2). We have explained the profile of the small hump, which is summarized in section 3.1.

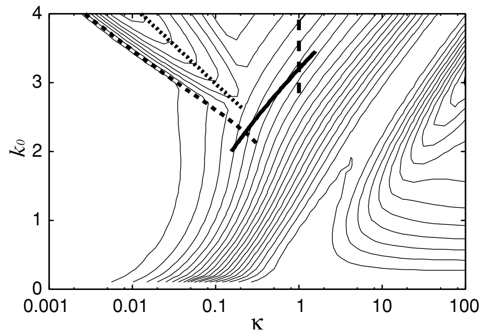

Figure 10. Approximation of the parameter region for the small hump of T_1 ((——) contour lines), trimmed by three thick curves: the normalized energy of the frontier soliton, estimated by equations (9) and (10), is equal to 0.1 (——). $\kappa = 1$ (– –), i.e., quasi-localization can exist for $\kappa \ll 1$. The normalized energy of the reflected waves is equal to 0.3 (– – –). The thick dotted curve denotes the approximant for the position of the small hump given by equation (14) with $\theta_{\mathrm{harm}} = 7\pi/4$, i.e., the intermediate value between the thick solid and dashed curves in figure 9.

4.4. Effect of increase in N

The above analysis is mainly based on the numerical results for $N = 2$. For $N \geqslant 3$, the parameter region for the delay in the soliton transmission can also be explained by the above analysis. Here, we describe two changes of properties of the small hump with the increase in N.

First, for $N \geqslant 4$, other small humps of T_1 appear near the first small hump. From the investigation on the temporal evolution of the wave, we found that these small humps appear due to two sources: (1) the second resonance between two springs and (2) reflected waves inside the harmonic lattice. With the increase in N, the small humps due to each contribution are clearly divided. With the further increase in N, these small humps shrink.

Second, for $N \gtrsim 10$, the top of the first small hump shifts to larger κ with the increase in N. This fact can be explained as follows: due to the dispersion of the harmonic lattice, the amplitude of the pulse at $n = N$ decreases with the increase in N. As a result, the energy density and α in equation (14) decrease. Therefore, the top of the small hump shifts to larger κ.

5. Summary and conclusion

We have numerically investigated the transmission of a soliton between two Toda lattices connected by a harmonic lattice using the soliton transmission coefficient T_1 as a measure of transmission.

For $N \geqslant 2$, large k_0 ($\gtrsim$2.5) and small κ ($\lesssim$1), we found that T_1 has a local maximum as a function of κ, where the delay in the transmission of the soliton occurs. By considering the mechanism of the delay in the soliton transmission, we obtained the parameter region for the delay in the soliton transmission on the (k_0, κ) domain. The position of the local maximum of T_1 is determined by the condition that the time scale of the motion of the stretched Toda spring being 1.5–2 times longer than that of the harmonic spring. The ratio between the time scales of the motion of the two springs is important for the delay in the soliton transmission.

The delay in the soliton transmission can be applied to a delay element based on a soliton. For example, our numerical results can be tested by using the nonlinear LC circuit (described in [12]), and a delay element can be constructed by the LC circuit. It is well known that an optical fibre supports the propagation of solitons. It may be possible to extend our idea to the transmission of the optical soliton in the connected optical fibres and construct an optical soliton delay device.

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